

# Lecture notes: axiomatic definitions and bibliography

## 1 Combinatorial data

**1 Definition** (System of apartments point of view). Let  $(W, S)$  be a Coxeter system. A **building** of type  $(W, S)$  is a polysimplicial complex  $X$  together with a collection  $\mathcal{A}$  of subcomplexes  $\mathbb{A} \subset X$ , called **apartments**, satisfying the following axioms:

- (B0) every apartment  $\mathbb{A} \in \mathcal{A}$  is isomorphic as cellular complex to the Coxeter complex of type  $(W, S)$ ;
- (B1) for any two faces  $F_1$  and  $F_2$  of  $X$ , there exists an apartment  $\mathbb{A} \in \mathcal{A}$  containing both;
- (B2) for any pair of apartments  $\mathbb{A}_1$  and  $\mathbb{A}_2$  such that  $\mathbb{A}_1 \cap \mathbb{A}_2$  contains a chamber of  $\mathbb{A}_1$  there exists a cellular complex isomorphism  $\mathbb{A}_1 \xrightarrow{\cong} \mathbb{A}_2$  pointwise fixing the faces of  $\mathbb{A}_1 \cap \mathbb{A}_2$ .

**2 Definition** (Chamber complex point of view). Let  $(W, S)$  be a Coxeter system. A **building**  $X$  of type  $(W, S)$  is a set of chambers  $\mathfrak{C}$  together with a map  $\delta : \mathfrak{C} \times \mathfrak{C} \rightarrow W$ , called the  **$W$ -distance**, satisfying the following axioms:  $\forall x, y, z \in \mathfrak{C}$

- (C1)  $\delta(x, y) = 1 \iff x = y$ ;
- (C2) if  $\delta(x, y) = w \in W$  and  $\delta(y, z) = s \in S$ , then  $\delta(x, z) \in \{w, sw\}$ ;  
moreover, if  $l(ws) > l(w)$ , then  $\delta(x, z) = ws$ ;
- (C3) for  $w = \delta(x, y)$  and  $s \in S$  there exists  $z \in \mathfrak{C}$  such that  $\delta(y, z) = s$  and  $\delta(x, z) = ws$ .

**3 Definition.** Let  $V$  be an  $\mathbb{R}$ -vector space and  $\Phi$  a subset of  $V$ . Then  $\Phi$  is a **root system** of  $V$  if it satisfies the following axioms:

- (RS1)  $\Phi$  is finite,  $0 \notin \Phi$  and  $\Phi$  spans  $V$ ;
- (RS2)  $\forall \alpha \in \Phi, \exists \alpha^\vee \in V^*, \alpha^\vee(\alpha) = 2$  and  $s_{\alpha, \alpha^\vee}(\Phi) = \Phi$ ,  
where  $s_{\alpha, \alpha^\vee}(x) = x - \alpha^\vee(x)\alpha$ ;
- (RS3)  $\forall \alpha \in \Phi, \alpha^\vee(\Phi) \subset \mathbb{Z}$ .

## 2 Groups data

**4 Definition.** A **Tits system** is the datum of a quadruple  $(G, B, N, S)$  consisting of an abstract group  $G$ , two subgroups  $B$  and  $N$  of  $G$  and a set  $S \subset W$ , where  $T = B \cap N$  and  $W = N/T$ , satisfying the following axioms:

- (TS1)  $G$  is generated by  $B \cup N$  and  $T = B \cap N$  is a normal subgroup of  $N$ ;
- (TS2)  $S$  generates  $W$  and consists in order 2 elements;
- (TS3) for any  $s \in S$  and  $w \in W$ , we have  $sBw \subset BwB \cup BswB$ ;
- (TS4) for any  $s \in S$ , we have  $sBs \neq B$ .

**5 Definition.** Let  $G$  be an abstract group and  $\Phi$  a root system. A **root groups datum** of  $G$  of type  $\Phi$  is the datum  $(T, (U_a, M_a)_{a \in \Phi})$  of subsets of  $G$  satisfying the following axioms:

- (RGD1)  $T$  is a subgroup of  $G$  and, for any root  $a \in \Phi$ , the set  $U_a$  is a nontrivial subgroup of  $G$ , called the root group of  $G$  associated to  $a$ ;
- (RGD2) for any roots  $a, b \in \Phi$ , the commutator group  $[U_a, U_b]$  is contained in the group generated by the root groups of the form  $U_{ra+sb}$  where  $r, s \in \mathbb{N}^*$  and  $ra + sb \in \Phi$ ;
- (RGD3) if  $a$  is a multipliable root, then  $U_{2a} \subset U_a$  and  $U_{2a} \neq U_a$ ;
- (RGD4) for any root  $a \in \Phi$ , the set  $M_a$  is a right coset of  $T$  in  $G$  and we have  $U_{-a} \setminus \{1\} \subset U_a M_a U_a$ ;
- (RGD5) for any roots  $a, b \in \Phi$  and any  $n \in M_a$ , we have  $n U_b n^{-1} = U_{s_a(b)}$  where  $s_a \in W(\Phi)$  is the orthogonal reflection with respect to  $a^\perp$  in the Weyl group  $W(\Phi)$  of  $\Phi$ ;
- (RGD6) for any ordering  $\Phi^+$  of  $\Phi$ , by setting  $\Phi^- = -\Phi^+ = \Phi \setminus \Phi^+$ , we have  $T U_{\Phi^+} \cap U_{\Phi^-} = \{1\}$ .

**6 Definition.** Let  $G$  be an abstract group and  $\Phi$  a root system. A **valued root groups datum** of  $G$  of type  $\Phi$  is the datum of a root groups datum  $(T, (U_a, M_a)_{a \in \Phi})$  together with maps  $\varphi_a : U_a \rightarrow \mathbb{R} \cup \{\infty\}$  for every roots  $a \in \Phi$ , satisfying the following axioms:

- (VRGD 0) for any root  $a \in \Phi$ , the image of the map  $\varphi_a$  contains at least 3 elements;
- (VRGD 1) for any root  $a \in \Phi$  and any element  $l \in \mathbb{R} \cup \{\infty\}$ , the preimage  $U_{a,l} = \varphi_a^{-1}([l; \infty])$  is a subgroup of  $U_a$  and group  $U_{a,\infty} = \{1\}$ ;
- (VRGD 2) for any root  $a \in \Phi$  and any element  $m \in M_a$ , the map  $u \mapsto \varphi_{-a}(u) - \varphi_a(mu m^{-1})$  is constant over  $U_{-a} \setminus \{1\}$ ;
- (VRGD 3) for any roots  $a, b \in \Phi$  such that  $b \notin -\mathbb{R}_+ a$  and any elements  $l, l' \in \mathbb{R}$ , the commutator subgroup  $[U_{a,l}, U_{b,l'}]$  is contained in the subgroup of  $G$  generated by the subgroups of the form  $U_{ra+sb, rl+sl'}$  for  $r, s \in \mathbb{N}^*$  and  $ra + sb \in \Phi$ ;
- (VRGD 4) for any multipliable root  $a \in \Phi$ , the map  $\varphi_{2a}$  is the restriction of the map  $2\varphi_a$  to  $U_{2a}$ ;
- (VRGD 5) for any root  $a \in \Phi$ , any element  $u \in U_a$  and any elements  $u', u'' \in U_{-a}$  such that  $u' u u'' \in M_a$ , we have  $\varphi_{-a}(u') = -\varphi_a(u)$ .

**7 Definition.** Let  $G$  be a reductive  $k$ -group that is split over a Galois extension  $\tilde{k}/k$  with Galois group  $\Sigma$ . Let  $\tilde{\Phi}$  be the absolute root system of  $G$  over  $\tilde{k}$ . A  $\tilde{k}$ -**system** of  $(G, \tilde{k}, k)$  is the datum of  $\tilde{k}$ -group isomorphisms:  $\tilde{x}_\alpha : \mathbb{G}_{a, \tilde{k}} \rightarrow U_\alpha$ . It is a  $\tilde{k}$ -**Chevalley system** if it satisfies the two first axioms:

- (CS1) for any absolute root  $\alpha \in \tilde{\Phi}$ , the isomorphisms  $\tilde{x}_\alpha$  and  $\tilde{x}_{-\alpha}$  are associated, that means there exists a homomorphism  $\theta_\alpha : \mathrm{SL}_{2, \tilde{k}} \rightarrow G$  such that for any  $u \in \tilde{k}$  we have  $\theta_\alpha \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \tilde{x}_\alpha(u)$  and  $\theta_\alpha \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} = \tilde{x}_{-\alpha}(u)$ ;
- (CS2) for any pair  $(\alpha, \beta)$  of absolute roots, there exists a sign  $\varepsilon_{\alpha, \beta} \in \{-1, 1\}$ , that depends only on  $\alpha$  and  $\beta$ , such that for any  $u \in \tilde{k}$ , we have  $\tilde{x}_{s_\alpha(\beta)}(u) = m_\alpha \cdot \tilde{x}_\beta(\varepsilon_{\alpha, \beta} u) \cdot m_\alpha^{-1}$ , where  $m_\alpha = \tilde{x}_\alpha(1) \cdot \tilde{x}_{-\alpha}(1) \cdot \tilde{x}_\alpha(1)$ .

It is a  **$\tilde{k}$ -Chevalley-Steinberg system** if, moreover, it satisfies the following axioms:

- (CS3) for any absolute root  $\alpha \in \tilde{\Phi}$  such that the relative root  $\alpha|_S \in \Phi$  is nondivisible, for any element  $\sigma \in \Sigma$ , we have the equality  $\tilde{x}_{\sigma(\alpha)} = \sigma \circ \tilde{x}_\alpha \circ \sigma^{-1}$ ;
- (CS4) for any absolute root  $\alpha \in \tilde{\Phi}$  such that the relative root  $\alpha|_S \in \Phi$  is divisible, for any element  $\sigma \in \Sigma$ , there exists a sign  $\varepsilon = \varepsilon_{\alpha,\sigma} \in \{-1, 1\}$  such that for any  $u \in \tilde{k}$  we have the equality  $\tilde{x}_{\sigma(\alpha)}(u) = \sigma(\tilde{x}_\alpha(\varepsilon\sigma^{-1}(u)))$ ;  
 if, moreover,  $\{\beta, \beta'\}$  is a subset of 2 distinct elements of  $\tilde{\Phi}$  such that  $\alpha = \beta + \beta'$  and  $\alpha|_S = 2a$  where  $a = \beta|_S = \beta'|_S \in \Phi$ , if  $\sigma$  fixes  $L_\alpha$ , then  $\varepsilon = -1$  if and only if  $\sigma$  is a nontrivial automorphism of  $L_\beta$ .

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